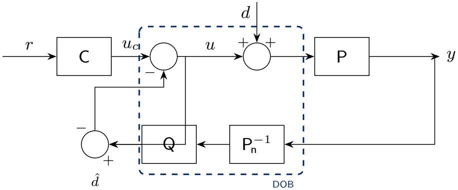
What a Classical DOB Does



- Estimate the "equivalent disturbance" via $\hat{d} = \mathsf{Q}\cdot\mathsf{P}_\mathsf{n}^{-1} y \; - \; \mathsf{Q}\cdot u$
- Subtract $\hat{d}$ from the control input
- Where $\mathsf{Q}(j\omega) \approx 1$ (low frequencies), the output *approximately* recovers nominal behaviour:

$$y(j\omega) \;\approx\; \frac{\mathsf{P_n\,C}}{1 + \mathsf{P_n\,C}}\, r(j\omega)$$

Why Attenuation $\neq$ Rejection

The disturbance-to-output transfer function is

$$T_d(s) \;=\; \frac{\mathsf{P}\,(1-\mathsf{Q})\,\mathsf{P_n}}{\mathsf{P}\,(\mathsf{Q}+\mathsf{P_n}\mathsf{C})+(1-\mathsf{Q})\,\mathsf{P_n}}\,.$$

- Q is a *low-pass filter*:
  $\mathsf{Q}(j\omega) \approx 1$ for small ω, but $\mathsf{Q} \neq 1$ exactly.
- So $(1-\mathsf{Q}) \neq 0 \implies T_d \neq 0$
- For a **persistent sinusoid** $d(t) = A\sin(\sigma t + \phi)$:

  a nonzero $T_d(j\sigma)$ means a *nonzero steady-state residual* — forever.

Placeholder:
$|T_d(j\omega)|$ sketch
nonzero at $\omega = \sigma$

Attenuation is not cancellation. Can we do better?

The Key Idea: Embed the Disturbance Model

Internal Model Principle (Francis & Wonham, 1976)

To reject a class of disturbances *exactly*, the controller must contain a copy of the dynamics that generate them.

Classical DOB

- Low-pass Q $\longrightarrow$ no disturbance model
- Attenuates, never cancels

Internal-Model DOB (this work)

- Q_q *embeds* the disturbance's minimal polynomial
- Asymptotic rejection: $e(t) \to 0$

The formal framework is *output regulation*;
we borrow exactly one ingredient — the internal model — and wire it into a DOB.

Problem Setup: Plant & Disturbance

Uncertain SISO plant (mechanical positioning system):

$$\mathsf{P}(s) = \frac{1}{Ms^2 + Bs}, \qquad \mathsf{P}_\mathsf{n}(s) = \frac{1}{M_n\, s^2 + B_n\, s}$$

- Mass M and damping B are *uncertain*; nominal values M_n, B_n are known.
- Input gain $g \in [g_{\min},\, g_{\max}]$ — bounded but unknown.

Disturbance generated by an *exosystem*:

$$\dot{w} = S\,w, \qquad d = R\,w$$

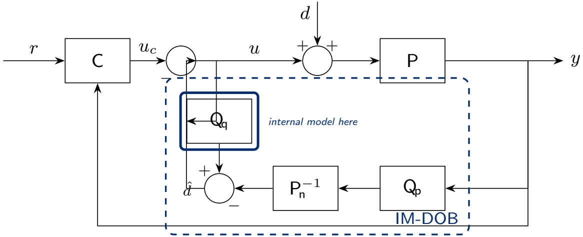
- The minimal polynomial $\Delta_\mathsf{w}(s)$ of S is **known** (i.e. the disturbance *frequencies* are known).
- The amplitude and phase — encoded in R and $w(0)$ — are **unknown**.

Two Disturbance Scenarios

	Constant sinusoid	**Growing sinusoid**
Signal	$d(t) = A\sin(\sigma t + \phi)$	$d(t) = A\,t\sin(\sigma t + \phi)$
Minimal poly.	$\Delta_{\mathsf{w}}(s) = s^2 + \sigma^2$	$\Delta_{\mathsf{w}}(s) = (s^2 + \sigma^2)^2$
Order μ	2	4

- **Same framework**, same filter design procedure.
- The growing case $t \cdot \sin(\sigma t)$ is a disturbance that *no finite-gain feedback* (including PID) can reject — but the internal model can.

Architecture: The Internal-Model DOB



- Q_p: conventional low-pass (implements P_n^{-1} causally)
- Q_q: internal-model filter — encodes $\Delta_w(s)$
- $\hat{d} = Q_p P_n^{-1} y - Q_q u$ (same DOB topology, different Q)

Where the Internal Model Lives: the $\mathsf{Q_q}$ Filter

Define the *feedforward filter* $\mathsf{F}(s) = \dfrac{1}{1 - \mathsf{Q_q}(s)}$, realized as

$$\dot{\xi} = \Phi\,\xi + J\,v, \qquad u = H\xi + v,$$

where Φ is the companion matrix of $\Delta_{\mathsf{w}}(s)$.

Key fact

The closed-loop filter matrix $A_q + B_q C_q$ has characteristic polynomial

$$\det\bigl(sI - (A_q + B_q C_q)\bigr) \;=\; \Delta_{\mathsf{w}}(s) \;=\; s^{\mu} + a_{\mu-1}s^{\mu-1} + \cdots + a_0\,.$$

The $\mathsf{Q_q}$ filter automatically contains an internal model of the exosystem $\dot{w} = Sw$.

Choosing the Filter: One Free Polynomial

$$\mathsf{Q_q}(s) \;=\; \frac{\gamma_{\mu-1}\, s^{\mu-1} + \cdots + \gamma_0}{s^{\mu} + \alpha_{\mu-1}\, s^{\mu-1} + \cdots + \alpha_0}$$

Design recipe:

1. Pick *any* Hurwitz polynomial for the denominator:
   $s^{\mu} + \alpha_{\mu-1} s^{\mu-1} + \cdots + \alpha_0$ (e.g. $(s+1)^{\mu}$, Butterworth, ...)
2. The numerator coefficients are *determined*:

$$\gamma_i \;=\; \alpha_i - a_i, \qquad i = 0, \ldots, \mu-1$$

   where a_i are the coefficients of $\Delta_{\mathsf{w}}(s)$.

- No Vandermonde matrix to invert
  (unlike prior work — those methods break when two disturbance frequencies coincide).
- No additional observer or filter blocks needed.

Main Result — In Words

If the combined filter matrix $\mathcal{A}_f$ is Hurwitz

and the time-scale parameter τ is small enough,

the tracking error converges to zero

— exactly, not merely bounded —

for every admissible plant uncertainty

and every disturbance whose frequency is modelled.

Main Result — Formal Statement

Theorem (Awai, Kim & Park, IFAC WC 2026)

Let:

- (A1) Plant parameters (ϕ, g) lie in a known compact set; minimal polynomial $\Delta_{\mathsf{w}}(s)$ of S is known.
- (A2) Nominal closed-loop matrix $\mathcal{A}_s$ is Hurwitz.
- (A3) Combined fast-subsystem matrix $\mathcal{A}_f$ is Hurwitz.

Then $\exists\, \tau^\star > 0$ such that for all $0 < \tau < \tau^\star$:

$$\lim_{t\to\infty} e(t) = 0 \qquad \text{and} \qquad \lim_{t\to\infty} \|q(t) - \Sigma\, w(t)\| = 0$$

for all admissible (ϕ, g), all $w(0)$, and all (R, S) consistent with Δ_{w}.

- $q(t) \to \Sigma w(t)$: the filter state *locks on* to the exosystem — this is the internal model at work.

The Lyapunov Certificate

Singular perturbation decomposition:

$$\underbrace{(x,\,\zeta)}_{\text{slow: plant + controller}} \qquad\qquad \underbrace{(\hat{p},\,\hat{q})}_{\text{fast: } \mathsf{Q_p}+\mathsf{Q_q}}$$

$$\dot{\chi} = \mathcal{A}_s\,\chi + \cdots\,, \qquad \tau\,\dot{\xi} = \mathcal{A}_f\,\xi + \cdots$$

Composite Lyapunov function:

$$V(\chi,\xi) \;=\; \chi^\top \mathcal{P}_s\,\chi \;+\; \xi^\top \mathcal{P}_f\,\xi$$

where $\mathcal{A}_s^\top \mathcal{P}_s + \mathcal{P}_s \mathcal{A}_s = -I$ and $\mathcal{A}_f^\top \mathcal{P}_f + \mathcal{P}_f \mathcal{A}_f = -I$.

Guarantee

For $0 < \tau < \tau^\star$: $\qquad \dot{V} \;\leq\; -\tfrac{1}{2}\|\chi\|^2 \;-\; \tfrac{1}{2\tau}\|\xi\|^2$

The existence of this V is the stability guarantee.
No need to walk through the inequalities.

The Design Knob: τ

- τ sets the **time-scale separation** between the controller and the DOB filters.
- Smaller τ
  $\Rightarrow$ faster observer & disturbance estimator
  $\Rightarrow$ better transient rejection
- But the Lyapunov guarantee requires

$$0 < \tau < \tau^\star$$

- $\tau^\star$ depends on plant uncertainty bounds and the filter polynomials (α_i, β_i).

Placeholder:
response overlay for
$\tau = 0.1,\ 0.05,\ 0.01$

τ is a single engineering knob with a computable safety limit.

FAQ: “Isn’t This Just Integral Action / PID?”

Integral action is the internal model of a *constant*:

$$\Delta_{\mathsf{w}}(s) = s \qquad \longleftrightarrow \qquad d(t) = \text{const.}$$

This work generalises to *any* modelled frequency:

$$\Delta_{\mathsf{w}}(s) = s^2 + \sigma^2 \quad \text{or} \quad (s^2 + \sigma^2)^2 \quad \text{or} \quad \ldots$$

Method	**Rejects**
PID	constant offset
This	$\sin(\sigma t)$
This	$t \cdot \sin(\sigma t)$

- A PID controller *cannot* reject $d(t) = t \sin(\sigma t)$ — no finite integral gain suffices.
- The internal-model DOB handles it by embedding $(s^2 + \sigma^2)^2$.

FAQ: "What If the Frequency σ Is Unknown?"

Adaptive-DOB extension

- Replace the fixed coefficients a_i in $\Delta_{\mathrm{w}}(s)$ with *online-estimated* coefficients $\hat{a}_i(t)$.
- The filter structure is identical; only the parameters adapt.
- **Status:** current work — stay tuned.

For today's results, σ is known.
The amplitude and phase remain unknown — and are handled.